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CONVERGENCE RATES FOR TIKHONOV REGULARIZATION OF NONLINEAR ILL-POSED PROBLEMS BASED ON DISTANCE FUNCTIONS

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INTRODUCTION

We study operator equations

$$F(x) = y \tag{1}$$

that characterize inverse problems with non-linear and continuous forward operators $F : D(F) \subset X \to Y$ and domain $D(F)$ mapping between infinite dimensional Hilbert spaces X and Y. Such problems frequently occur in applied inverse problems. To obtain stable approximate solutions of (1) regularization methods are required when instead of exact data y from the range $\mathcal{R}(F)$ of F noisy data $y^\delta \in Y$ with

$$\|y - y^\delta\| \le \delta \tag{2}$$

and noise level $\delta > 0$ are given. We assume attainability for exact data y, i.e., the existence of some $x^\dagger \in D(F)$ with $F(x^\dagger) = y$. Under some further assumptions as Fréchet differentiability in $x^\dagger$ we apply nonlinear Tikhonov regularization.

CONVERGENCE RARTES UNDER GENERAL SOURCE CONDITIONS

Using index functions $\psi(t)$ one can assume general source conditions of the form

$$x^\dagger - x^* = \psi\left(F'(x^\dagger)^* F'(x^\dagger)\right) v, \quad (v \in X), \tag{3}$$

which allow the formulation of convergence rates for the regularized solutions. In the talk we will study such convergence rate results for general index functions ψ in (3) similar to [4] as well as for some special source conditions of power-type and logarithmic type.

ON THE INTERPLAY OF SOLUTION SMOOTHNESS AND SMOOTHING PROPERTIES OF THE LINEARIZED FORWARD OPERATOR

For linear ill-posed problems we characterize the a priori information on the 'smoothness' (in a very extended sense) of the solution by a bounded self-adjoint linear operator $G : X \to X$ with non-closed range. For our nonlinear problem (1) we therefore assume

$$x^\dagger - x^* = Gw \qquad (w \in X) \tag{4}$$

With respect to their nature in the modelling process of equation (1) the operator G and the forward operator F, respectively its linearization $F'(x^\dagger)$ at the solution point $x^\dagger$, are in principle independent. We need to link those conditons. A suitable way of doing that was used in [1] applying range inclusions

$$\mathcal{R}(\rho(G)) \subset \mathcal{R}(\varphi(F'(x^\dagger)^* F'(x^\dagger))) \tag{5}$$

with index functions ρ and φ.

The so-called benchmark function $\varphi(t)$ expresses an upper limit of the expected smoothness of the difference $x^\dagger - x^*$ with

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respect to the operator $F'(x^\dagger)$ by the assumption

$$x^\dagger - x^* \notin \mathcal{R}(\varphi(F'(x^\dagger)^* F'(x^\dagger)). \qquad (6)$$

This means that the distance function

$$d_\varphi(R) := \inf\left\{ \left\| x^\dagger - x^* - \varphi(F'(x^\dagger)^* F'(x^\dagger))u \right\| : u \in X, \|u\| \le R \right\}, \qquad (7)$$

measuring the violation of the required source condition, is strictly positive for all $R \ge 0$, and under our assumptions it tends to zero as $R \to \infty$. This concept was, for example, introduced in [3] for linear operator equations. An example of explicit verification of such a distance function was presented in [2]. The condition (6) means that a general source condition (3) can only be true for some index function if the decay rate of $\varphi(t) \to 0$ as $t \to 0$ is faster than the corresponding rate of ψ. There exist results that special decay rates of distance functions yield convergence rates in the linear case. We will demonstrate similar results for our context.

RANGE INCLUSIONS YIELDING GENERAL SOURCE CONDITIONS

We are going to have a look at general source conditions (3) based on the solution smoothness (4) and the range inclusion (5). As a consequence we then can show convergence rates for the nonlinear Tikhonov regularization without the explicit knowledge and use of a function ψ in the general source condition. Instead we require the knowledge of an index function ρ in the range inclusion characterizing the interplay of $F'(x^\dagger)^* F'(x^\dagger)$ and G.

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